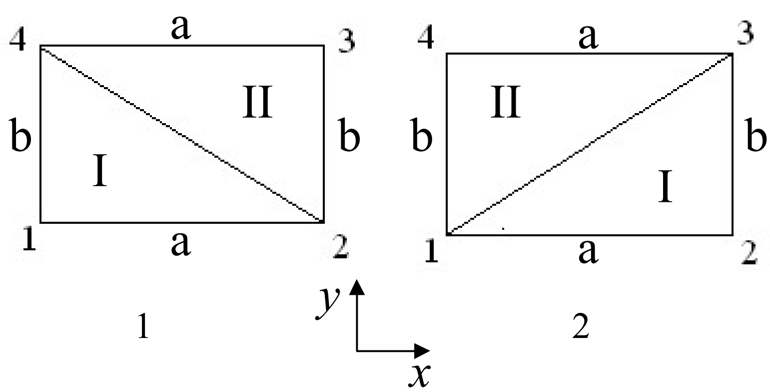
The procedure for performing the calculation work on the basics of the FE-method

2013

Full Name	NR	NZ	NA	NB	NH	NF1	NF2	NP1	NP2

1. Calculation scheme number NR

Numbers (1, 2, 3, 4) it is node sequence and node numbers. The center of the coordinate system at node 1.



2. Fixation option NZ

NZ	1	2	3	4	5	6
	1, 2	1, 3	1, 4	2, 3	2, 4	3, 4

3. Width a: 0,1·NA (m)

4. Height b: 0,1·NB (m)

5. Thickness h: 0,002·NH (m)

6. The application point of the load vector NF1

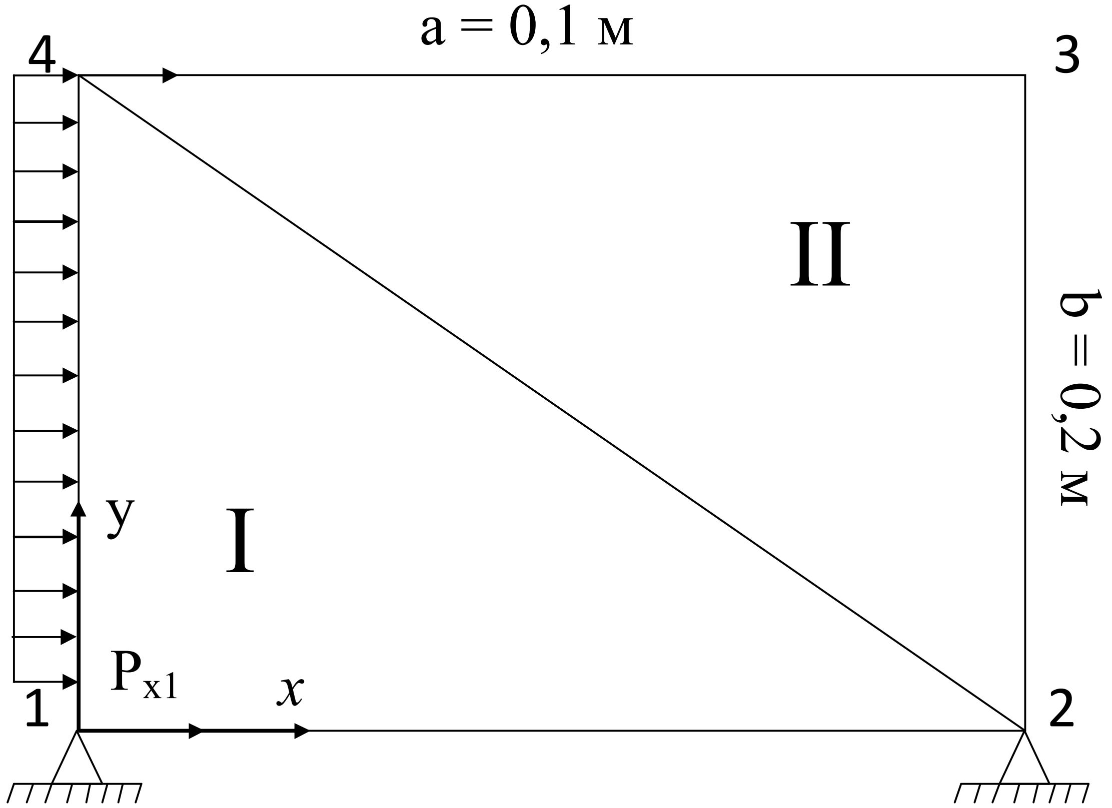
7. Value of the load vector 5000·NF2 (N)

8. Action line of linear (dispersed) load (p_{vc}) NP1

NP1	1	2	3	4	5	6
v, *c*	1, 2	1, 3	1, 4	2, 3	2, 4	3, 4

9. Value of the linear load NP2
p_{vc} = 40000·NP2 (N/m)

The value of linear load on each node is equal $P_x = p_{vc} \cdot b/2$.
Load values are rounded to the nearest whole (P=1049,95 =1050 N)



Formation of global displacement vector and load vector

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q_{x3} \\ q_{y3} \\ q_{x4} \\ q_{y4} \end{Bmatrix}, \begin{Bmatrix} N_{x1} \\ N_{y1} \\ N_{x2} \\ N_{y2} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} P_{x1} + N_{x1} \\ N_{y1} \\ N_{x2} \\ N_{y2} \\ 0 \\ P_{y3} \\ P_{x4} \\ 0 \end{Bmatrix}$$

Formation of stiffness matrices of each FE

Area of each finite element is equal $S_1 = S_2 = a \cdot b/2$.

Material matrix $[D]$

$$[D] = \frac{E}{1-\mu^2}\begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{pmatrix}$$

Node coordinates :

$$\begin{matrix} x_1 = 0 & y_1 = 0 \\ x_2 = a & y_2 = 0 \\ x_3 = a & y_3 = b \\ x_4 = 0 & y_4 = b \end{matrix}$$

Geometric matrix of FE $[B_1]$ or $[B_2]$

$$[B] = \frac{1}{2S}\begin{pmatrix} y_j - y_k & 0 & y_k - y_i & 0 & y_i - y_j & 0 \\ 0 & x_k - x_j & 0 & x_i - x_k & 0 & x_j - x_i \\ x_k - x_j & y_j - y_k & x_i - x_k & y_k - y_i & x_j - x_i & y_i - y_j \end{pmatrix},$$

i, j, k – nodes of FE.

The numbering of nodes should increase counterclockwise from minimum to maximum. For example, FE №I: 1(i)-2(j)-4(k), FE №2: 2(i)-3(j)-4(k).

FE №	i	j	k
I	1	2	4
II	2	3	4

$$[B_1] = \frac{1}{2S_1}\begin{pmatrix} y_2 - y_4 & 0 & y_4 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_4 - x_2 & 0 & x_1 - x_4 & 0 & x_2 - x_1 \\ x_4 - x_2 & y_2 - y_4 & x_1 - x_4 & y_4 - y_1 & x_2 - x_1 & y_1 - y_2 \end{pmatrix}$$

$$[B_2] = \frac{1}{2S_2}\begin{pmatrix} y_3 - y_4 & 0 & y_4 - y_2 & 0 & y_2 - y_3 & 0 \\ 0 & x_4 - x_3 & 0 & x_2 - x_4 & 0 & x_3 - x_2 \\ x_4 - x_3 & y_3 - y_4 & x_2 - x_4 & y_4 - y_2 & x_3 - x_2 & y_2 - y_3 \end{pmatrix}$$

The factor $1/(2S_i)$ is moved into the matrix by multiplying all the coefficients of the matrix by it.

Transposed matrix $[B_1]$.

$$[B_1^T] = \frac{1}{2S_1}\begin{pmatrix} y_2 - y_4 & 0 & x_4 - x_2 \\ 0 & x_4 - x_2 & y_2 - y_4 \\ y_4 - y_1 & 0 & x_1 - x_4 \\ 0 & x_1 - x_4 & y_4 - y_1 \\ y_1 - y_2 & 0 & x_2 - x_1 \\ 0 & x_2 - x_1 & y_1 - y_2 \end{pmatrix}$$

Stiffness matrix:

$$[K_1] = S_1 h [B_1{}^T][D][B_1]$$

Matrices must be multiplied exactly in the order in which they are written!

The resulting stiffness matrix of the FE №I :

$$[K_1] = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & k_{15}^1 & k_{16}^1 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & k_{25}^1 & k_{26}^1 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 & k_{34}^1 & k_{35}^1 & k_{36}^1 \\ k_{41}^1 & k_{42}^1 & k_{43}^1 & k_{44}^1 & k_{45}^1 & k_{46}^1 \\ k_{51}^1 & k_{52}^1 & k_{53}^1 & k_{54}^1 & k_{55}^1 & k_{56}^1 \\ k_{61}^1 & k_{62}^1 & k_{63}^1 & k_{64}^1 & k_{65}^1 & k_{66}^1 \end{pmatrix},$$

Formation of major stiffness matrix and resolving system of equations

$$\begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & 0 & 0 & k_{17} & k_{18} \\ k_{21} & k_{22} & k_{23} & k_{24} & 0 & 0 & k_{27} & k_{28} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\ 0 & 0 & k_{53} & k_{54} & k_{55} & k_{56} & k_{57} & k_{58} \\ 0 & 0 & k_{63} & k_{64} & k_{65} & k_{66} & k_{67} & k_{68} \\ k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} \\ k_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88} \end{pmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q_{x3} \\ q_{y3} \\ q_{x4} \\ q_{y4} \end{Bmatrix} = \begin{Bmatrix} P_{x1} + N_{x1} \\ N_{y1} \\ N_{x2} \\ N_{y2} \\ 0 \\ P_{y3} \\ P_{x4} \\ 0 \end{Bmatrix}$$

i-i	*i-j*	*i-k*
j-i	*j-j*	*j-k*
k-i	*k-j*	*k-k*

FE №	*i*	*j*	*k*
I	1	2	4
II	2	3	4

Then for the matrices $[K_1]$ и $[K_2]$ the tables will be written in the form

1	1-1	1-2	1-4	2-2	2-3	2-4	2
2	2-1	2-2	2-4	3-2	3-3	3-4	3
4	4-1	4-2	4-4	4-2	4-3	4-4	4
	1	2	4	2	3	4	

The table for major stiffness matrix $[K]$

1	2	3	4	
1	1	0	1	1
1	x	2	x	2
0	2	2	2	3
1	x	2	x	4

Determination of unknown displacements and reactions in nodes

Multiplying the matrix [K] by the vector {q} according to the rules (the rows of the first matrix are multiplied by the columns of the second matrix), we get

$$k_{55} \cdot q_{x3} + k_{56} \cdot q_{y3} + k_{57} \cdot q_{x4} + k_{58} \cdot q_{y4} = 0$$
$$k_{65} \cdot q_{x3} + k_{66} \cdot q_{y3} + k_{67} \cdot q_{x4} + k_{68} \cdot q_{y4} = P_{y3}$$
$$k_{75} \cdot q_{x3} + k_{76} \cdot q_{y3} + k_{77} \cdot q_{x4} + k_{78} \cdot q_{y4} = P_{x4}$$
$$k_{85} \cdot q_{x3} + k_{86} \cdot q_{y3} + k_{87} \cdot q_{x4} + k_{88} \cdot q_{y4} = 0$$

A system of four equations with four unknowns q can be solved by Cramer's method.

Then, from the first four equations, we determine the unknown reactions N_{x1}, N_{y1}, N_{x2}, N_{y2}.

Example of calculation

a = 0,1 m; b = 0,2 m; h = 0,003 m; $E = 2\cdot10^{11}$ Pa; μ = 0,33;
P_{y3} = 10000 N (node 3); p = 80000 N/m (line 1-4); nodes 1 and 2 are fixed.

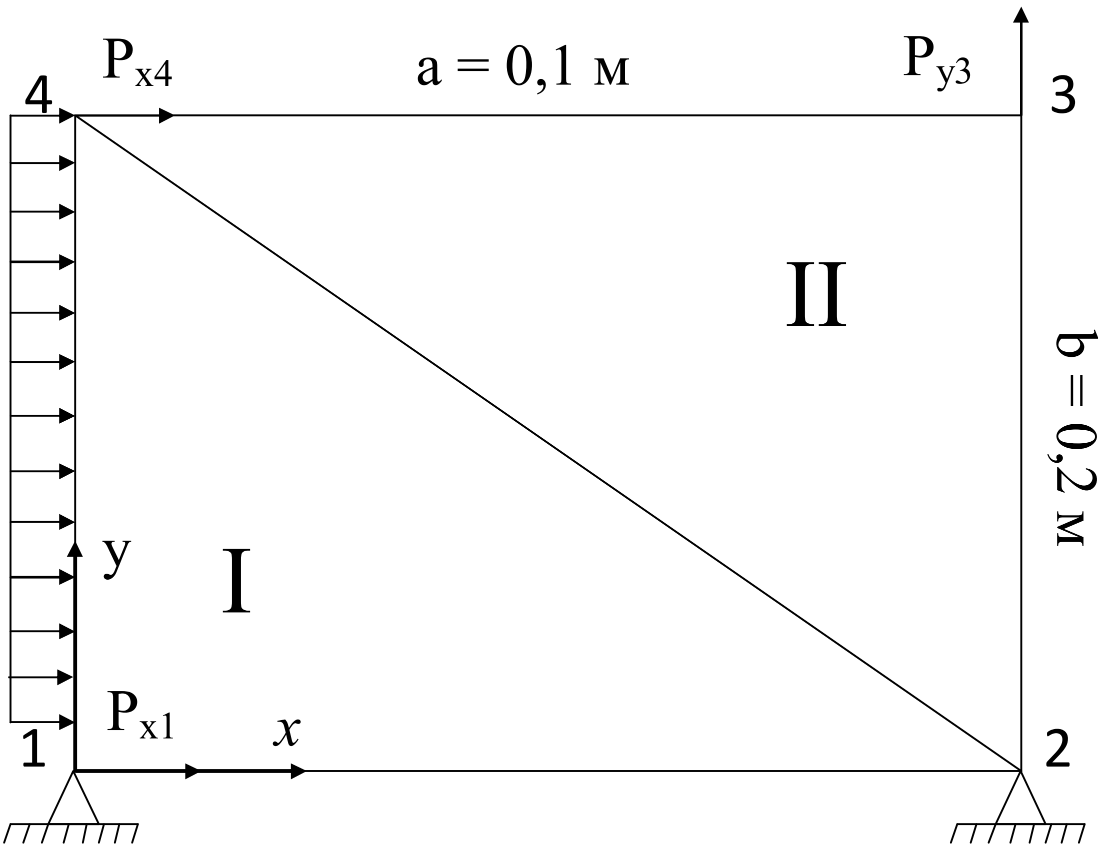


Fig. 1 - Calculation scheme

The global displacement vector and load vector:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q_{x3} \\ q_{y3} \\ q_{x4} \\ q_{y4} \end{Bmatrix}, \begin{Bmatrix} 8000 + N_{x1} \\ N_{y1} \\ N_{x2} \\ N_{y2} \\ 0 \\ 10000 \\ 8000 \\ 0 \end{Bmatrix}$$

Areas of the FE: $S_1 = S_2 = 0{,}01$ m^2.

Materal matrix [D]:

$$[D] = \frac{2\cdot 10^{11}}{1-0{,}33^2}\begin{pmatrix} 1 & 0{,}33 & 0 \\ 0{,}33 & 1 & 0 \\ 0 & 0 & \dfrac{1-0{,}33}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2{,}244\cdot 10^{11} & 7{,}407\cdot 10^{10} & 0 \\ 7{,}407\cdot 10^{10} & 2{,}244\cdot 10^{11} & 0 \\ 0 & 0 & 7{,}519\cdot 10^{10} \end{pmatrix}$$

Node coordinates :

$$x_1 = 0 \quad y_1 = 0$$
$$x_2 = 0{,}1 \quad y_2 = 0$$
$$x_3 = 0{,}1 \quad y_3 = 0{,}2$$
$$x_4 = 0 \quad y_4 = 0{,}2$$

Numbering of nodes FE .

FE№	*i*	*j*	*k*
I	1	2	4
II	2	3	4

Geometric matrices $[B_1]$ и $[B_2]$

$$[B_1] = \begin{pmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 5 \\ -5 & -10 & 0 & 10 & 5 & 0 \end{pmatrix}$$

$$[B_2] = \begin{pmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -5 & 0 & 5 & 0 & 0 \\ -5 & 0 & 5 & 10 & 0 & -10 \end{pmatrix}$$

Transposed matrices B_1 and B_2.

$$[B_1^T] = \begin{pmatrix} -10 & 0 & -5 \\ 0 & -5 & -10 \\ 10 & 0 & 0 \\ 0 & 0 & 10 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix}; \; [B_2^T] = \begin{pmatrix} 0 & 0 & -5 \\ 0 & -5 & 0 \\ 10 & 0 & 5 \\ 0 & 5 & 10 \\ -10 & 0 & 5 \\ 0 & 0 & -10 \end{pmatrix}$$

Stiffness matrices K_1 and K_2:

$$[K_1] = S_1 h [B_1^T][D][B_1]$$

$$= \begin{pmatrix} 7{,}2972\cdot 10^8 & 2{,}2388\cdot 10^8 & -6{,}7333\cdot 10^8 & -1{,}1278\cdot 10^8 & -5{,}6391\cdot 10^7 & -1{,}1110\cdot 10^8 \\ 2{,}2388\cdot 10^8 & 3{,}9390\cdot 10^8 & -1{,}1110\cdot 10^8 & -2{,}2556\cdot 10^8 & -1{,}1278\cdot 10^8 & -1{,}6833\cdot 10^8 \\ -6{,}7333\cdot 10^8 & -1{,}1110\cdot 10^8 & 6{,}7333\cdot 10^8 & 0 & 0 & 1{,}1110\cdot 10^8 \\ -1{,}1278\cdot 10^8 & -2{,}2556\cdot 10^8 & 0 & 2{,}2556\cdot 10^8 & 1{,}1278\cdot 10^8 & 0 \\ -5{,}6391\cdot 10^7 & -1{,}1278\cdot 10^8 & 0 & 1{,}1278\cdot 10^8 & 5{,}6391\cdot 10^7 & 0 \\ -1{,}1110\cdot 10^8 & -1{,}6833\cdot 10^8 & 1{,}1110\cdot 10^8 & 0 & 0 & 1{,}6833\cdot 10^8 \end{pmatrix}$$

$$[K_2] = S_2 h [B_2^T][D][B_2]$$

$$= \begin{pmatrix} 5{,}6391\cdot 10^7 & 0 & -5{,}6391\cdot 10^7 & -1{,}1278\cdot 10^8 & 0 & 1{,}1278\cdot 10^8 \\ 0 & 1{,}6833\cdot 10^8 & -1{,}1110\cdot 10^8 & -1{,}6833\cdot 10^8 & 1{,}1110\cdot 10^8 & 0 \\ -5{,}6391\cdot 10^7 & -1{,}1110\cdot 10^8 & 7{,}2972\cdot 10^8 & 2{,}2388\cdot 10^8 & -6{,}7333\cdot 10^8 & -1{,}1278\cdot 10^8 \\ -1{,}1278\cdot 10^8 & -1{,}6833\cdot 10^8 & 2{,}2388\cdot 10^8 & 3{,}9390\cdot 10^8 & -1{,}1110\cdot 10^8 & -2{,}2556\cdot 10^8 \\ 0 & 1{,}1110\cdot 10^8 & -6{,}7333\cdot 10^8 & -1{,}1110\cdot 10^8 & 6{,}7333\cdot 10^8 & 0 \\ 1{,}1278\cdot 10^8 & 0 & -1{,}1278\cdot 10^8 & -2{,}2556\cdot 10^8 & 0 & 2{,}2556\cdot 10^8 \end{pmatrix}$$

Major stiffness matrix [K]

$$\begin{pmatrix} 7{,}2972\cdot10^8 & 2{,}2388\cdot10^8 & -6{,}7333\cdot10^8 & -1{,}1278\cdot10^8 & 0 & 0 & -5{,}6391\cdot10^7 & -1{,}1110\cdot10^8 \\ 2{,}2388\cdot10^8 & 3{,}9390\cdot10^8 & -1{,}1110\cdot10^8 & -2{,}2556\cdot10^8 & 0 & 0 & -1{,}1278\cdot10^8 & -1{,}6833\cdot10^8 \\ -6{,}7333\cdot10^8 & -1{,}1110\cdot10^8 & 7{,}2972\cdot10^8 & 0 & -5{,}6391\cdot10^7 & -1{,}1278\cdot10^8 & 0 & 2{,}2388\cdot10^8 \\ -1{,}1278\cdot10^8 & -2{,}2556\cdot10^8 & 0 & 3{,}9390\cdot10^8 & -1{,}1110\cdot10^8 & -1{,}6833\cdot10^8 & 2{,}2388\cdot10^8 & 0 \\ 0 & 0 & -5{,}6391\cdot10^7 & -1{,}1110\cdot10^8 & 7{,}2972\cdot10^8 & 2{,}2388\cdot10^8 & -6{,}7333\cdot10^8 & -1{,}1110\cdot10^8 \\ 0 & 0 & -1{,}1278\cdot10^8 & -1{,}6833\cdot10^8 & 2{,}2388\cdot10^8 & 3{,}9390\cdot10^8 & -1{,}1110\cdot10^8 & -2{,}2556\cdot10^8 \\ -5{,}6391\cdot10^7 & -1{,}1278\cdot10^8 & 0 & 2{,}2388\cdot10^8 & -6{,}7333\cdot10^8 & -1{,}1110\cdot10^8 & 7{,}2972\cdot10^8 & 0 \\ -1{,}1110\cdot10^8 & -1{,}6833\cdot10^8 & 2{,}2388\cdot10^8 & 0 & -1{,}1278\cdot10^8 & -2{,}2556\cdot10^8 & 0 & 3{,}9390\cdot10^8 \end{pmatrix}$$

Since the displacements of nodes 1 and 2 are equal to zero, then

$$\begin{pmatrix} 7{,}2972\cdot10^8 & 2{,}2388\cdot10^8 & -6{,}733\cdot10^8 & -1{,}1110\cdot10^8 \\ 2{,}2388\cdot10^8 & 3{,}9390\cdot10^8 & -1{,}1110\cdot10^8 & -2{,}2556\cdot10^8 \\ -6{,}7333\cdot10^8 & -1{,}1110\cdot10^8 & 7{,}2972\cdot10^8 & 0 \\ -1{,}1278\cdot10^8 & -2{,}2556\cdot10^8 & 0 & 3{,}9390\cdot10^8 \end{pmatrix} \cdot \begin{Bmatrix} q_{x3} \\ q_{y3} \\ q_{x4} \\ q_{y4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10000 \\ 8000 \\ 0 \end{Bmatrix}$$

Having compiled a system of linear equations and using Cramer's method, we find the roots:

$$\begin{pmatrix} q_{x3} = 7{,}622\cdot10^{-5}\ \text{м} & q_{y3} = 2{,}78\cdot10^{-5}\ \text{м} \\ q_{x4} = 8{,}553\cdot10^{-5}\ \text{м} & q_{x3} = 3{,}775\cdot10^{-5}\ \text{м} \end{pmatrix}$$

Substituting the found values of the displacements of nodes 3 and 4 into the resolving system of equations, we calculate the reactions in nodes 1 and 2 :

$$\begin{pmatrix} N_{x1} = -17017\ H & N_{y1} = -16000\ H \\ N_{x2} = 1016{,}6\ H & N_{y2} = 6000\ H \end{pmatrix}$$

An example of solving a system of equations by Cramer's method by MathCAD

Запишем выделенные из общих матриц фрагменты матрицы сил и матрицы жесткостей:

$$Qp := \begin{pmatrix} 0 \\ 10000 \\ 8000 \\ 0 \end{pmatrix} \qquad K := \begin{pmatrix} 7.297210^{8} & 2.238810^{8} & -6.733310^{8} & -1.111010^{8} \\ 2.238810^{8} & 3.939010^{8} & -1.111010^{8} & -2.255610^{8} \\ -6.733310^{8} & -1.111010^{8} & 7.297210^{8} & 0 \\ -1.127810^{8} & -2.255610^{8} & 0 & 3.939010^{8} \end{pmatrix}$$

Для составления определителя необходимо выделить из полученной матрицы К отдельные столбцы в виде переменных, где верхний индекс обозначает номер столбца, нижний - номер строки. Причем первая строка и первый столбец имеют индексы "ноль":

$$J1 := K^{\langle 0 \rangle} = \begin{pmatrix} 7.2972\times 10^{8} \\ 2.2388\times 10^{8} \\ -6.7333\times 10^{8} \\ -1.1278\times 10^{8} \end{pmatrix} \qquad J2 := K^{\langle 1 \rangle} \qquad J3 := K^{\langle 2 \rangle} \qquad J4 := K^{\langle 3 \rangle}$$

Составим и решим матрицу-определитель, согласно методу Крамера:

$$\Delta := \left| \begin{pmatrix} J1_0 & J2_0 & J3_0 & J4_0 \\ J1_1 & J2_1 & J3_1 & J4_1 \\ J1_2 & J2_2 & J3_2 & J4_2 \\ J1_3 & J2_3 & J3_3 & J4_3 \end{pmatrix} \right| = 4.51484\times 10^{33}$$

$$\Delta 1 := \left| \begin{pmatrix} Qp_0 & J2_0 & J3_0 & J4_0 \\ Qp_1 & J2_1 & J3_1 & J4_1 \\ Qp_2 & J2_2 & J3_2 & J4_2 \\ Qp_3 & J2_3 & J3_3 & J4_3 \end{pmatrix} \right| = 3.39628\times 10^{29} \qquad \Delta 2 := \left| \begin{pmatrix} J1_0 & Qp_0 & J3_0 & J4_0 \\ J1_1 & Qp_1 & J3_1 & J4_1 \\ J1_2 & Qp_2 & J3_2 & J4_2 \\ J1_3 & Qp_3 & J3_3 & J4_3 \end{pmatrix} \right| = 1.26551\times 10^{29}$$

$$\Delta 3 := \left| \begin{pmatrix} J1_0 & J2_0 & Qp_0 & J4_0 \\ J1_1 & J2_1 & Qp_1 & J4_1 \\ J1_2 & J2_2 & Qp_2 & J4_2 \\ J1_3 & J2_3 & Qp_3 & J4_3 \end{pmatrix} \right| = 3.82147\times 10^{29} \qquad \Delta 4 := \left| \begin{pmatrix} J1_0 & J2_0 & J3_0 & Qp_0 \\ J1_1 & J2_1 & J3_1 & Qp_1 \\ J1_2 & J2_2 & J3_2 & Qp_2 \\ J1_3 & J2_3 & J3_3 & Qp_3 \end{pmatrix} \right| = 1.69708\times 10^{29}$$

Определим перемещения в узлах в направлении осей х и у, м:

$$qx3 := \frac{\Delta 1}{\Delta} = 7.52248\times 10^{-5} \qquad qy3 := \frac{\Delta 2}{\Delta} = 2.803\times 10^{-5}$$

$$qx4 := \frac{\Delta 3}{\Delta} = 8.46424\times 10^{-5} \qquad qy4 := \frac{\Delta 4}{\Delta} = 3.7589\times 10^{-5}$$